MTH 151 (36 points) Fall 2022, Exam 1

Name: ______________________________ #(   ) Oct 6, 2022

To receive full credit, you must show all of your work. The final answer without steps/work is only worth 0.25 of the points of the question

1. Find the anti-derivative

   (a) (2 points) $\int 4\sec^2 x + \dfrac{4}{1+x^2}\,dx$

   $4\tan x + 4\tan^{-1}(x) + C$

   (b) (2 points) $\int \dfrac{x^2+x}{x^2}\,dx = \int \dfrac{x^2}{x^2} + \dfrac{x}{x^2}\,dx = \int 1 + \dfrac{1}{x}\,dx$

   $= x + \ln|x| + C.$

s' $S(t)$ $\int v(t)$
$v(t)$
$s'' = v'$ $a(t)$ $\int a(t)$

2. The acceleration function of a body moving along a coordinate line is

$$a(t) = -\cos t - \sin t, \quad t \geq 0.$$

   (a) (2 points) Find its velocity function at any time t if it is located at the origin and has an initial velocity of 4 m/sec (i.e. $v(0) = 4$).

   $v(t) - v(0) = \int_0^t -\cos x - \sin x\,dx = -\sin x + \cos x \Big|_0^t$

   $v(t) - 4 = (-\sin t + \cos t) - (-\sin 0 + \cos 0)$

   $\boxed{v(t) = -\sin t + \cos t + 3}$ $t \geq 0$

   (b) (2 points) Find the displacement of the body as a function of time.

   $\text{displacement} = \int_0^t v(x)\,dx = \int_0^t -\sin +$

   $= \cos x + \sin x + 3x \Big|_0^t = (\cos t + \sin t + 3t) - (\cos 0 + \sin 0 + 3(0))$

   $\boxed{s(t) = \text{disp.} = \cos t + \sin t + 3t - 1}$

3. Evaluate the following integral

(a) (3 points) $\int \frac{e^x}{1+e^x}\,dx$

$u = 1+e^x \qquad du = e^x\,dx$

$= \int \frac{du}{u} = \ln|u| + C$

$= \ln|1+e^x| + C.$

(b) (5 points) $\int_e^{e^2} \frac{1}{x\ln x}\,dx$

$u = \ln x \longrightarrow du = \frac{1}{x}\,dx$

$x = e \Rightarrow u = \ln e = 1$

$x = e^2 \Rightarrow u = \ln e^2 = 2$

$\int_{x=e}^{x=e^2} \frac{1}{x\ln x}\,dx = \int_1^2 \frac{1}{u}\,du = \ln|u|\Big|_1^2$

$= \ln|2| - \cancel{\ln|1|}^{\,0}$

$= \ln 2.$

or

$\int \frac{1}{x\ln x}\,dx = \ln|\ln|x||$

$\int_e^{e^2} \frac{1}{x\ln x}\,dx = \ln|\ln|x||\Big|_e^{e^2} = \ln|\ln|e^2|| - \ln|\cancel{\ln|e|}|$

$= \ln 2 - \cancel{\ln 1}\ 0$

$= \ln 2$

4. (5 points) Consider the function $f(x) = e^{\cos(t)}\sin(t)$ on the interval $[0, \pi/2]$. Find the average value, f_{ave}, of the function f on the given interval. (Give the exact answer. NO approximation!).

$$f_{ave} = \frac{1}{b-a}\int_a^b f(t)\, dt$$

$u = \cos t$

$du = -\sin t\, dt$

$t = 0 \to u = 1$

$t = \frac{\pi}{2} \to u = 0$

$$= \frac{1}{\frac{\pi}{2} - 0}\int_{t=0}^{t=\pi/2} e^{\cos t}\sin(t)\, dt$$

$$= \frac{2}{\pi}\int_{u=1}^{u=0} e^u \cdot (-du)$$

$$= \frac{2}{\pi}\int_{u=0}^{u=1} e^u\, du = \frac{2}{\pi}\cdot e^u\Big|_0^1$$

$$= \frac{2}{\pi}\left(e^1 - e^0\right) = \frac{2}{\pi}(e-1)$$

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

or

$$\int e^{\cos t}\sin t\, dt = \int e^u \cdot (-du) = -e^u + C = -e^{\cos t} + C$$

then

$$\int_0^{\pi/2} e^{\cos t}\sin t\, dt = -e^{\cos t}\Big|_0^{\pi/2} = -\left(e^{\cos \pi/2} - e^{\cos 0}\right)$$

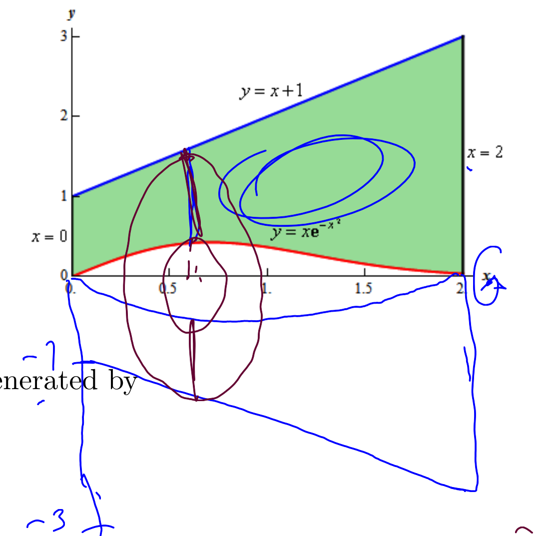
$$= -\left(e^0 - e^1\right) = -(1-e) = e-1$$

So,

$$f_{ave} = \frac{1}{b-a}\int_a^b f(t)\, dt = \frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2} e^{\cos t}\sin t\, dt = \frac{2}{\pi}(e-1)$$

5. A region R is bounded by the curves $y = xe^{-x^2}$, $y = x + 1$, $x = 2$, and the $y-$axis. **Set up the integral(s)**

(a) (3 points) needed to find the area of the region R. (**Don't integrate**).



y_{top} y_{bot}

$$A(x) = \int_0^2 (x+1) - (x e^{-x^2})\, dx$$

(b) (3 points) needed to find the **volume** of the solid generated by revolving R about the $x-$axis. (**Don't integrate**).

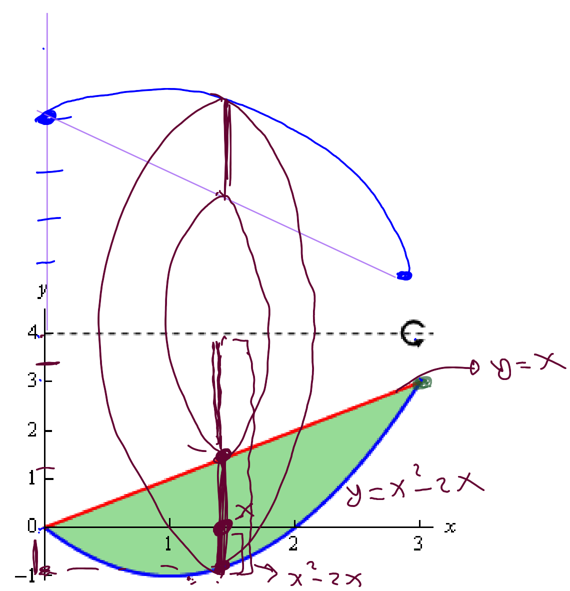
$$V = \int_0^2 \pi(x+1)^2 - \pi\left(xe^{-x^2}\right)^2 dx$$

$A(x) = \pi(r_1)^2 - \pi(r_2)^2$

6. (4 points) Find the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$. about the line $y = 4$.

$$V = \int_0^3 A(x)\, dx$$

$$= \int_0^3 \pi\left(x^2 - 2x + 4\right)^2 - \pi\left(4 - x\right)^2 dx$$

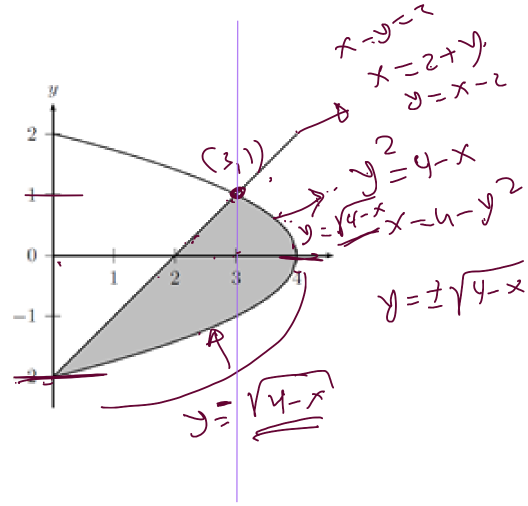


7. **Set up the integral(s)** needed to find the area of the region bounded by $y^2 = 4 - x$ and $x - y = 2$.

   (a) (2 points) using integration with respect to y
   (**Don't integrate**)

$$A(y) = \int x_{right} - x_{left} \; dy$$

$$= \int_{-2}^{1} (4-y^2) - (2+y) \; dy$$



$$A(x) = \int y_{top} - y_{bot} \; dx$$

$$= \int_0^3 (y-2) - \left(-\sqrt{4-x}\right) dx + \int_3^4 \left(\sqrt{4-x}\right) - \left(-\sqrt{4-x}\right) dx$$

   (b) (3 points) using integration with respect to x.
   (**Don't integrate**)

$A(x) =$